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LETTER TO THE EDITOR

Bifurfcating solutions of the topologically massive gauge theories with external sources

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Abstract. We demonstrate how bifurcating solutions for the $(2+1)$-dimensional topologically massive Yang–Mills gauge field equations with external sources can be constructed.

Recently there have been many interests in constructing classical solutions of $(2+1)$-dimensional field theories involving the Chern–Simons (CS) term. Since for the CS action alone the classical solution is trivial, the Yang–Mills (YM) action [1] or the charged scalar field terms [2] or both [3] are usually incorporated. Classical solutions are useful as they may provide some insight into the full quantized theory. In this letter, we show how bifurcating solutions to the topologically massive gauge field theory in the presence of an external source can be constructed. Solutions for topologically massive gauge fields with external sources have been discussed in [4]; however, no bifurcation is exhibited. Bifurcating solutions in $(3+1)$-dimensional YM theories were first presented by Jackiw *et al* [5]: two branches emanating from the same point in the plot of energy H against total charge Q were found. Our approach in searching for bifurcating solutions follows that of [6].

For the SU(2) gauge group, the YM equations with the CS term in $(2+1)$-dimensional spacetime and the Bianchi identity are respectively

$$D_\mu F^{\mu\nu} + \xi \tilde{F}^\nu = J^\nu \tag{1a}$$

$$D_\nu \tilde{F}^\nu = 0 \qquad \tilde{F}^\nu \equiv \tfrac{1}{2}\varepsilon^{\nu\alpha\beta} F_{\alpha\beta} \tag{1b}$$

where J^ν is the external source and the metric is $g_{\mu\nu} = (-++)$. The coefficient ξ of the CS term is replaced by $-\mathrm{i}\xi$ in Euclidean spacetime. As usual, the choice of ansatz for the gauge field A^a_μ is of utmost importance and we employ the following:

$$A^a_j = (\phi^a P + \delta^a_3/\rho)\phi_j + n^a n_j T \tag{2a}$$

$$A^a_0 = \phi^a V \tag{2b}$$

where $\rho = (x_1^2 + x_2^2)^{1/2}$, $n^j = x^j/\rho$, $\phi^a = \varepsilon^{aj} n^j$ and $\phi^3 = n^3 = 0$; the group index $a = 1, 2, 3$ and the space index $j = 1, 2$. The functions, P, V and T depend on ρ only. For a given prescribed external source, it is extremely difficult to construct the solution for the gauge field. Instead the technique of [7] is adopted; one first obtains the gauge potential A with the desired properties and then from which the external source current is derived. On substituting the ansatz (2) into (1), we have

$$-V''-V'/\rho+VT^2=\xi(P'+P/\rho)+J^{0a}\phi^a \tag{3a}$$

$$2V'T+VT'+VT/\rho=-\xi TP+J^{0a}\delta_3^a \tag{3b}$$

$$-(TP)'-(P'+P/\rho)T=\xi VT+J^{ja}\delta_3^a\phi_j \tag{3c}$$

$$P''+P'/\rho+P/\rho^2-PT^2=-\xi V'+J^{ja}\phi^a\phi_j \tag{3d}$$

$$(V^2-P^2)T=J^{ia}n^a n_i \tag{3e}$$

where the prime indicates differentiation with respect to ρ. To simplify these equations we choose the source current as follows:

$$J_0^a=(I-VT^2)\phi^a+M\delta_3^a \tag{4a}$$

$$J_i^a=T(V^2-P^2)n^a n_i-(T^2P\phi^a+(TP)'\delta_3^a)\phi_i \tag{4b}$$

where I and M are functions of ρ only, and gauge covariant conservation of the external source current, $D_\mu J^\mu=0$, is ensured. This choice reduces equations (3) tremendously:

$$-V''-V'/\rho=\xi(P'+P/\rho)-I \tag{5a}$$

$$-(P'+P/\rho)=\xi V \tag{5b}$$

$$2V'T+VT'+VT/\rho=-\xi TP-M. \tag{5c}$$

A solution for equations (5a), (5b) can be constructed:

$$V=K(\alpha+2-sz) \tag{6a}$$

$$P=-zK \tag{6b}$$

$$I=\xi^2K[s^2(3\alpha+5)-s(s^2-1)z-(\alpha+2)-s(3\alpha^2+7\alpha+3)/z+\alpha^2(\alpha+2)/z^2] \tag{6c}$$

where $K\equiv z^\alpha\,\mathrm{e}^{-sz}$, $z\equiv\xi\rho$ and α, s are parameters for the charge distribution. To solve for the remaining equation (5c), we expand the functions T and M by power series and after some manipulation, we find

$$T=\xi L \qquad L\equiv z^\beta\,\mathrm{e}^{-tz} \tag{6d}$$

$$M=\xi^2 zKL\{2s^2+st-1-[s(4\alpha+\beta+7)+t(\alpha+2)]z^{-1}+(\alpha+2)(2\alpha+\beta+1)z^{-2}\} \tag{6e}$$

where β and t are parameters for the charge distribution.

For our solution (6), the expressions for the total energy H and the total charge Q can be straightforwardly evaluated after some lengthy computation. The energy [7] is calculated from the energy-momentum tensor $T^{\mu\nu}$,

$$H=\int \mathrm{d}^2x\,T^{00}$$
$$=\int \mathrm{d}^2x[\tfrac{1}{2}(E_i^aE^{ai}+B^aB^a)+J_i^aA^{ai}] \tag{7}$$

where $E_i^a\equiv F_{i0}^a$ and $B^a=\frac{1}{2}\varepsilon_{ij}F^{aij}$ are respectively the non-Abelian electric and magnetic fields. We find

$$H=(as^2+b)s^{-2(\alpha+1)}+(cs^2+ds+e)(s+t)^{-2(\alpha+\beta+2)} \tag{8a}$$

where

$$a=\Gamma_1[2(6\alpha^2+16\alpha+9)(2\alpha+1)-3(2\alpha+3)(2\alpha+1)(\alpha+1)$$
$$-8\alpha(2\alpha+3)(\alpha+2)+4\alpha(\alpha+2)^2] \quad (8b)$$

$$b=\Gamma_1[(2\alpha+3)(2\alpha+1)(\alpha+1)-4(\alpha+1)(\alpha+2)(2\alpha+1)+2(2\alpha+1)(\alpha+2)^2] \quad (8c)$$

$$c=\Gamma_2[(2\alpha+2\beta+3)(\alpha+\beta+1)-4(\alpha+\beta+1)(\alpha+2)+2(\alpha+2)^2] \quad (8d)$$

$$d=-4\Gamma_2(\alpha+2)(\beta-1)t \quad (8e)$$

$$e=\Gamma_2[2t^2(\alpha+2)^2-(2\alpha+2\beta+3)(\alpha+\beta+1)] \quad (8f)$$

and

$$\Gamma_1 \equiv (\pi\alpha)4^{-(\alpha+1)}\Gamma(2\alpha)$$
$$\Gamma_2 \equiv (6\pi)4^{-(\alpha+\beta+2)}\Gamma(2\alpha+2\beta+2)$$

and $\alpha>0$, $\beta>-(\alpha+1)$. As for the total external charge strength, we project along the direction ϕ^a,

$$Q=\int d^2x\, J^{0a}\phi^a \quad (9)$$

and obtain

$$Q=4\pi(s+2t)^{-(\alpha+2\beta+3)}\Gamma(\alpha+2\beta+2)[t(\alpha+2)-\beta s]. \quad (10)$$

As mentioned earlier, bifurcation means the branching of the total energy $H(\lambda)$ of the gauge field and external source system when the total external charge $Q(\lambda)$ is

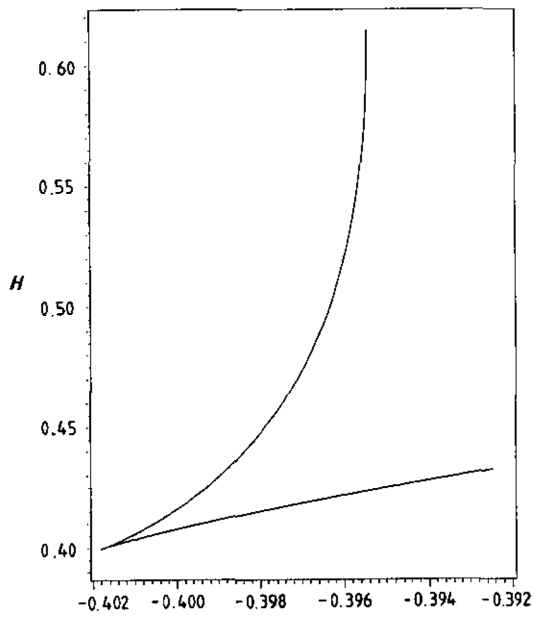

Figure 1. Plot of H against Q when the parameter α is varied but $\beta=1.000\,000$, $s=2.224\,259$ and $t=0.400\,000$.

varied [5], where λ is a set of parameters. The existence of local minima of $H(\lambda)$ and $Q(\lambda)$ at common parametric values, say $\lambda = \lambda_c$, will imply the bifurcation of the $H(Q)$ curve [6]. We have four parameters here, $\lambda = (\alpha, \beta, s, t)$, and it is not easy to find the common parametric values of α, β, s, t at which $H(\lambda)$ and $Q(\lambda)$ have their respective minima. However by trial and error, we find after much effort that H and Q do possess their respective local minimum when $\beta = 1.000\,000$, $t = 0.400\,000$, $s = 2.224\,259$ and $\alpha = 1.829\,673$. In our search for the bifurcation point, we fix the values of β, s and t each time, the parameter α is then continuously varied. In this way, H and Q essentially depend on only one parameter, namely α. In figure 1, we present the bifurcating curve with the characteristic cusp in the plot of H against Q, for which the values of the parameters β, t, s are fixed as above whilst the parameter α is varied. Note that the solutions P, T, V and the source functions I M can be easily plotted and are not shown here.

We end with short remarks.

(1) It is not difficult to construct solutions for equations (1), but if we require the solutions to have finite total energy H and total charge Q so that the branching occurs in the plot of H against Q, then much effort is demanded.

(2) There may be other common parametric values of (α, β, s, t) at which H and Q can have their respective local minima. A systematic search is possible.

(3) The energy expression (7) is gauge dependent because of the term $J_i^a A^{ai}$. Gauge-independent energy results if we set $J_i^a = 0$, but then construction of bifurcating solutions is harder.

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